

Scaling in the One-Dimensional Anderson Localization Problem in the Region of Fluctuation States

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(Received 5 July 2002; published 27 March 2003)

We numerically study the distribution function of the conductivity (transmission) in the one-dimensional tight-binding Anderson model in the region of fluctuation states. We show that while single parameter scaling in this region is not valid, the distribution can still be described within a scaling approach based upon the ratio of two fundamental quantities, the localization length, l_{loc} , and a new length, l_s , related to the integral density of states. In an intermediate interval of the system's length L , $l_{\text{loc}} \ll L \ll l_s$, the variance of the Lyapunov exponent does not follow the predictions of the central limit theorem, and may even grow with L .

DOI: 10.1103/PhysRevLett.90.126601

PACS numbers: 72.15.Rn, 41.20.Jb, 42.25.Bs

Introduction.—A coherent transport in mesoscopic disordered systems is characterized by strong fluctuations and the non-self-averaging nature of the transport coefficients such as conductance, g , or transmittance, T [1,2]. Therefore, a description of the transport in such systems requires dealing with entire distribution functions of the respective quantities. The scaling approach to the transport allows one to introduce a reduced “macroscopic” description of such distributions independent of microscopic details of the underlying Hamiltonians [3,4] with parameters of the distributions playing the role of the scaling variables [4]. If the entire distribution can be parametrized by a single parameter, the respective system is said to obey single parameter scaling (SPS). A quantity, which is most convenient to work with when describing the statistics of transport is the Lyapunov exponent (LE), $\tilde{\gamma}(L) = (1/2L) \ln(1 + 1/g)$, where L is the length of the system [4,5]. Finite size LE, $\tilde{\gamma}(L)$, is self-averaging quantity (it approaches a nonrandom limit γ , when $L \rightarrow \infty$ [2]), and its distribution approaches a Gaussian form for asymptotically long systems. The Gaussian distributions are characterized by two parameters, the mean value, γ , and the variance, σ^2 , and the SPS hypothesis suggests that they are related to each other in a universal way. Such a relation, which can be expressed in the form

$$\tau = \sigma^2 L l_{\text{loc}} = 1, \quad (1)$$

where $l_{\text{loc}} = 1/\gamma$ is identified with the localization length, was first conjectured by Anderson *et al.* [4] and reproduced later by many authors within the framework of the phase randomization hypothesis [2]. The phase randomization was proven rigorously for in-band states (those belonging to the spectrum of underlying ordered systems) for some one-dimensional models (the Anderson model [6] and a continuous model with a white-noise random potential [7]), as well as for some quasi-one-dimensional models [8]. For the Lloyd model, the authors of Ref. [9] showed that Eq. (1) (corrected by a factor of 2)

holds for in-band states even though the distribution of phases is not uniform.

At the same time, numerical results presented in Ref. [10] showed that Eq. (1) is not valid for fluctuation states arising due to disorder outside the original spectrum. A boundary between SPS and non-SPS spectral regions in the exactly solvable Lloyd model was shown to be determined by a relation $l_{\text{loc}}(E) \approx l_s(E)$ [9], where l_s is a new length, defined through the number of states, $N(E)$, per unit length, between E and the closest genuine spectral boundary:

$$l_s^{-1} = \sin[\pi N(E)]. \quad (2)$$

In the region of fluctuation states, when $N(E) \ll 1$, or $1 - N(E) \ll 1$, so that $l_s \gg l_{\text{loc}}$, SPS is not valid. Complimenting analytical calculations by numerical simulations, the authors of Ref. [9] showed that the criterion for SPS found for the Lloyd model is valid for other models as well.

Thus, it is clear that the problem of the statistics of transport in the region of fluctuation states requires a separate consideration. The distinction between this situation and the case of in-band states can be qualified as a difference between underbarrier tunneling and overbarrier scattering. In 3D this difference is clear: the latter case corresponds to the spectral region of extended states with the diffusive transport, while the former takes place in the region of localized states. The problem of the underbarrier tunneling in disordered systems was first considered in Ref. [5] (for reviews of subsequent papers see Refs. [11,12]). In a 1D situation all states are localized, and the transmission for all energies can be described as a resonant tunneling via rare transparent configurations [13]. Therefore, the difference between the two transport regimes is more subtle and was noticed only recently [9,10]. Correspondingly, while the case of the in-band states can be considered settled by the SPS theory, the properties of the distribution function of

conductance/transmittance for a pure one-dimensional case of underbarrier tunneling are studied very little. Besides obvious fundamental importance, an additional motivation to deal with this problem comes from the development of photonic band gap materials, in which a new type of fluctuation photonic states is possible [14]. These states form “Lifshits tails” in the band gaps of disordered photonic structures, and provide a unique opportunity to study resonant underbarrier tunneling with a scattering of light.

The main objective of the present paper is to study numerically the distribution function of LE in the region of fluctuation states using ideas of the scaling approach. The main question, which we seek to answer is the following: “Is the distribution function of the LE in the non-SPS region determined completely by microscopic details of the respective Hamiltonian, or can it still be described macroscopically in a universal manner?” We show that the distribution of conductance in this region, while not completely universal, still demonstrates surprising scaling properties. In particular, using Monte Carlo simulations for the one-dimensional tight-binding Anderson model with diagonal disorder, we find that for sufficiently long systems the function τ , introduced in Eq. (1), depends upon a single parameter, $\kappa = l_{\text{loc}}/l_s$. We also find strong deviations of the distribution function from the Gaussian form. However, the third moment turns out to have the same scaling behavior as the variance, indicating that despite the deviation of the distribution function from the Gaussian, it still can be parametrized by only two parameters, l_{loc} and l_s .

Model and technical details.—We consider the tight-binding model with a diagonal disorder, which is described by the following equations of motion:

$$\psi_{n+1} + \psi_{n-1} + (U_n - E)\psi_n = 0, \quad (3)$$

where random on-site energies U_n are described by a uniform probability distribution: $P(U_n) = 1/(2U)$ if $|U_n| < U$, and $P(U_n) = 0$ otherwise. LE is defined as

$$\gamma(E) = \lim_{N \rightarrow \infty} \frac{1}{N} \log \|T_N \cdots T_1\| = \lim_{L \rightarrow \infty} \tilde{\gamma}(L), \quad (4)$$

where T_k are transfer matrices

$$T_k = \begin{pmatrix} E - U_k & -1 \\ 1 & 0 \end{pmatrix}. \quad (5)$$

LE is calculated iteratively using Eq. (4) in a standard way [15]. To investigate the statistics of $\tilde{\gamma}(L)$ in systems with the finite length L , we keep L fixed while collecting statistics from about 120 000 realizations. The integral density of states for each realization was calculated using the phase formalism [2] and was averaged over all realizations. The resulting value was used to calculate the length l_s according to Eq. (2). Studying the dependence of the distribution of $\tilde{\gamma}$ on L we take care to have $L \gg l_{\text{loc}}$

for all strengths of the disordered potential, U , and values of energy, E . However, in the region of fluctuation states, where $l_{\text{loc}} < l_s$, it is possible to have $l_s > L \gg l_{\text{loc}}$. In this regime, which does not exist in the SPS region, the L dependence of the variance may be different from the standard behavior given by the central limit theorem. In order to verify this assumption, we considered systems with lengths satisfying both $L < l_s$ and $L > l_s$. When collecting statistics, we discarded all data corresponding to $l_{\text{loc}} < 5$ and $l_s > 1000L$. This way we ensured that our results were not influenced by nonrepresentative fluctuations and states localized over microscopical regions of the sample.

Results.—Figure 1 shows the dependence of the scaling function τ defined in Eq. (1) on κ in the asymptotic limit of very long systems, $L \gg l_s, l_{\text{loc}}$. Data used to generate this figure were obtained for different values of E and U , and one can see that they all fall nicely on the scaling curve $\tau(\kappa)$. For $\kappa > 1$, τ approaches its universal SPS value of unity while for smaller κ it steeply decreases. A similar result was obtained for a periodic-on-average model in Ref. [9], where the scaling function $\tau(\kappa)$ was originally proposed. Our results convincingly show that σ^2 can indeed be expressed in terms of the scaling function $\tau(\kappa)$ regardless of the microscopic nature of the model under consideration.

While the exact shape of the function $\tau(\kappa)$ varies slightly from model to model (compare to the results of Ref. [9]), the essential qualitative properties of τ seem to be quite universal. We are most interested in the region $\kappa < 1$, where τ demonstrates a sharp decrease. Analytical calculations carried out in Ref. [9] for the Lloyd model produced $\tau(\kappa) \approx \kappa$. Our results show, however, a much steeper decrease of τ . Also, in the model considered here,

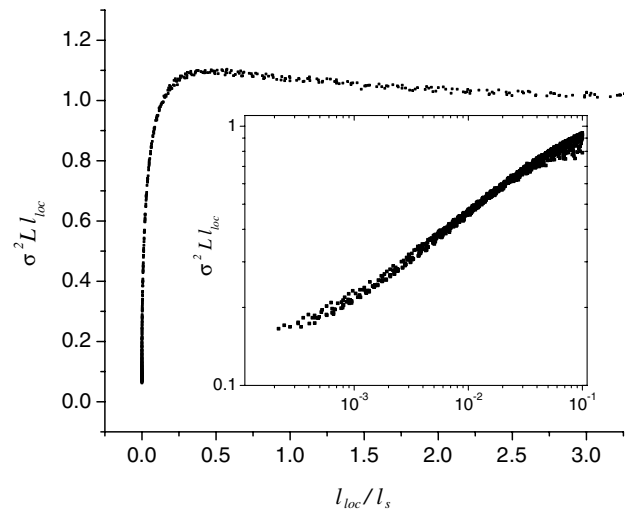


FIG. 1. Dependence of the scaling parameter τ on $\kappa = l_{\text{loc}}/l_s$ for a set of potentials ($U = 0.08 - 0.155$). In the inset dependence $\tau(\kappa)$ in the non-SPS region ($\kappa \ll 1$) is shown in log-log scale for $1 < L/l_s < 5$.

$\tau(\kappa)$ must remain nonzero for $\kappa = 0$. Indeed, $\kappa = 0$ corresponds to the exact genuine boundary of the spectrum of our system. Unlike the Lloyd model, where the spectrum boundary is at infinity, in our model the boundaries of the spectrum are at $E_b = \pm(2 + U)$. The variance of LE does not vanish at finite energies, and therefore $\tau_{\text{lim}} \equiv \tau(0)$ is not equal to zero.

In order to understand the behavior of τ at $\kappa \ll 1$ we conduct a detailed study of this region for systems with different L . Our results can be summarized in the following form:

$$\tau = C\kappa^\alpha + \tau_{\text{lim}}. \quad (6)$$

Replotting $\tau(\kappa)$ in log-log coordinates for $\kappa < 1$ (see inset of Fig. 1) we see that while κ changes by more than 2 orders of magnitude, the data form a good straight line with the exception of points corresponding to extremely small values of κ . This means that τ_{lim} can be neglected for most of the non-SPS region, and becomes significant only in the immediate vicinity of E_b . Using linear regression we can estimate parameters C and α for systems with different lengths. The results of the fit reveal that C and α are constants independent of any parameters of the system under consideration for $L > l_s$. This result confirms the one-parameter form of $\tau(\kappa)$ given by the first term of Eq. (6) for sufficiently long systems. The degree of universality of these coefficients still remains an open question requiring similar studies of other models. We can speculate, however, that it is likely that systems can be divided into several universality classes on the basis of the values of C and α .

For shorter systems with $L < l_s$, both C and α show additional dependence upon the length L ; see Fig. 2 where α and C are plotted versus L/l_s .

The results of this analysis lead to two important conclusions. First, the length l_s not only establishes the boundary between SPS and non-SPS regions of the spectrum, but also determines a crossover system length marking the transition to systems with a universal single parameter form for τ . Second, as it was anticipated, in the regime $l_{\text{loc}} \ll L < l_s$ the scaling of the variance of LE changes from the simple $1/L$ dependence to a more complicated form due to the dependence of α on L/l_s . We attempted to fit this dependence in the region of small L/l_s by several types of trial functions; the best fit was obtained with $\alpha(L/l_s) \sim \ln(l_s/L)$. With this assumption a new scaling for the variance becomes

$$\sigma^2 \propto \frac{1}{L l_{\text{loc}}} \exp[\alpha(L/l_s) \ln \kappa] \propto L^{-(1+\ln \kappa)}. \quad (7)$$

It is interesting to note that when κ decreases, $1 + \ln \kappa$ becomes negative and σ^2 starts growing with L in this interval of lengths. This behavior can be qualitatively understood from the following arguments: The condition

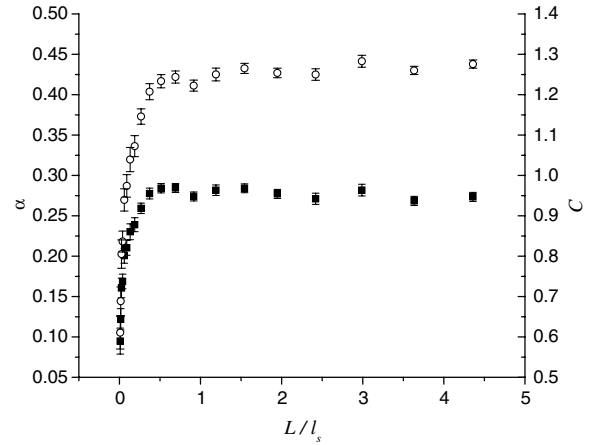


FIG. 2. Index of the scaling parameter α (filled squares, left axis) and factor C (circles, right axis) as functions of L/l_s .

$L \ll l_s$ means that for most of the realizations of the random potential no states exist in the energy interval under discussion. The transmission through such realizations fluctuates rather weakly. The greatest contribution to the transmission fluctuations give those few realizations that can support at least a single state. The probability for such realizations to arise grows when the length of the system increases, resulting in the respective increase of σ^2 .

This behavior, of course, breaks down for very large values of l_s , which correspond to states close to the genuine spectral boundary, because for these states σ^2 is determined by a nonuniversal correction to τ given by τ_{lim} . This limiting value can be found using the weak disorder expansion of Derrida *et al.* [16], which gives

$$\tau_{\text{lim}} \approx \frac{\sigma_U^2}{4\gamma_0 \sinh^2 \gamma_0} \propto \sqrt{U}, \quad (8)$$

where $\sigma_U^2 = U^2/3$ is the variance of the potential, γ_0 is LE in the gap region of the original ordered system, and we use the fact that $E_b = 2 + U$ for the system under consideration. One can see from this expression that τ_{lim} depends on microscopic characteristics of the original Hamiltonian. However, for a model with a Gaussian distribution of site energies, the genuine spectral boundary lies at infinity, where $\gamma_0 \rightarrow \infty$. The first of the equalities in Eq. (8), which can be applied to various distributions, gives in this case $\tau_{\text{lim}} = 0$. We can expect this to be true for all models with spectral boundaries at infinity. For this class of models, $\tau(\kappa)$ gives a completely universal, at least within a given model, description of the variance of LE.

In order to characterize deviations of the distribution function of LE from the Gaussian form, we also studied scaling properties of the third cumulant $\varrho = \langle (\gamma - \langle \gamma \rangle)^3 \rangle$, which describes the skewness of the distribution function. To analyze scaling properties of ϱ we

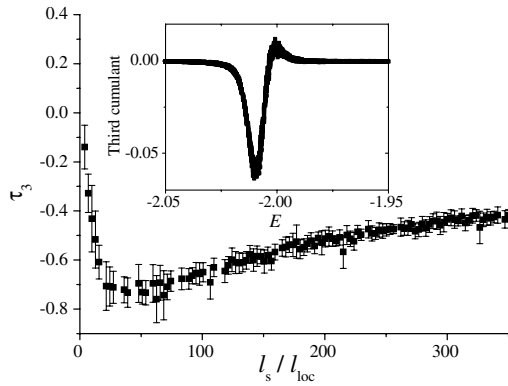


FIG. 3. Dependence of τ_3 on κ deeply inside of the non-SPS region ($\kappa \ll 1$). In the inset scaled third cumulant ϱL^2 is depicted as a function of the energy near the band edge for the potential $U = 0.05$ and sample lengths from 1040 to 2160.

introduced a function analogous to

$$\tau_3 = \varrho L^2 l_{\text{loc}}. \quad (9)$$

One can see from Fig. 3 that while data for the parameter τ_3 are rather noisy, it shows a relatively good scaling behavior as a function of the single parameter κ in the non-SPS region. This fact itself is quite remarkable since it demonstrates that even deviations from the Gaussian in the region of fluctuation states can be described within the scaling procedure suggested here. It is interesting to note that the sign of skewness changes not very far from the boundary between SPS and non-SPS regions of the spectrum. In the SPS region the absolute value of skewness decreases dramatically becoming essentially zero within the accuracy of our calculations. This is illustrated in the inset of Fig. 3, which represents the energy dependence of the third moment of the distribution. The skewness also decreases deeper in non-SPS regions where it becomes extremely small beyond the genuine spectral boundary. The quality of our raw data did not allow us to determine if τ_3 also depends upon L/l_s for shorter systems, but we expect that τ_3 behaves similar to τ in this regard.

Conclusion.—In this Letter we carried out a detailed study of the distribution function of conductance in the spectral region of fluctuation states. We showed that apart from a small nonuniversal contribution, which is important only in the immediate vicinity of the genuine spectral boundary, the conductance distribution in this region can be described using a simple scaling approach. Within this approach the variance is described by the scaling function τ , Eq. (1), and the third moment of the distribution is characterized by the function τ_3 , Eq. (9). For long enough systems both scaling functions depend on the single variable $\kappa = l_{\text{loc}}/l_s$. The presence of such a scaling behavior would be natural for the model with the white-

noise potential, because such a model has a natural scaling variable $E/\sigma_U^{3/2}$ [2], and our scaling parameter κ depends on only this variable [9]. Our numerical results showed, however, that the parameter κ provides a more universal description of the distribution function valid also outside of the white-noise model. While we considered here only the tight-binding model, we believe that our results qualitatively describe statistics of light transmission through band gaps of disordered photonic crystals. Experimental measurements of the transmission distribution in such systems can be used for verification of our results and as a method of measuring parameters l_{loc} and l_s .

The authors thank Steve Schwarz for reading and commenting on the manuscript. This work was supported by AFOSR under Contract No. F49620-02-1-0305, and partially by PSC-CUNY and CUNY collaborative grants.

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