Quantum Transport and Quantum Coherence (Localisation 2002)
J. Phys Soc. Jpn. Vol. **72** (2003) Suppl. A pp. 173–174

Scaling and Fluctuations of the Lyapunov Exponent in a 2D Anderson Localisation Problem

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(Received August 14, 2002)

KEYWORDS: localisation, Anderson model, fluctuation, scaling

In one-dimensional (1D) systems it is well known that the Lyapunov exponent (LE) has a normal distribution and that its average value $\langle \gamma_{1D} \rangle$ is related to its variance σ_{1D}^2 by

$$\frac{\sigma_{1D}^2 L}{\langle \gamma_{1D} \rangle} = 1. \tag{1}$$

Here $\langle \ldots \rangle$ represents a statistical average over realizations of the random potential. Expression (1) was first conjectured by Anderson *et al.*[1] and later derived by many different authors within the framework of the random phase approximation. A correct and rigorous criterion for the validity of (1) was established only much later.[2,3] For a sufficiently long 1D system the logarithm of the dimensionless conductance g is approximately

$$\ln g \simeq -2\gamma L. \tag{2}$$

Since a normal distribution is parameterised by its mean and variance, (1) establishes the single parameter scaling of the conductance distribution for 1D systems.

The objective of this paper is to establish a generalisation of (1) for the two dimensional Anderson model with diagonal disorder. We first investigate numerically the behaviour of the ratio on the l.h.s. of (1) in quasi-1D systems with length L and width M where $L \gg M$. (In this case, strictly speaking, we mean the smallest positive LE.) Contrary to 1D, and as we shall see below, this ratio is not a constant in quasi-1D systems. While at first sight this seems to call into question the validity of single parameter scaling, a careful analysis reveals that the ratio obeys a one parameter scaling law of the form

$$\frac{\sigma^2 L}{\langle \gamma_L \rangle} = F_\sigma \left(\frac{\xi}{M} \right). \tag{3}$$

where ξ is the localisation length in 2D limit. This makes it clear that a deviation from (1) does not, of itself, imply a breakdown of one parameter scaling.

We consider a two-dimensional Anderson model described by a tight binding Hamiltonian with diagonal disorder and nearest neighbour hopping on a square lattice.

$$H = \sum_i \epsilon_i c_i^\dagger c_i - \sum_{\langle i,j \rangle} c_i^\dagger c_j. \tag{4}$$

Site energies ϵ_i are uniformly distributed on the interval $[-W/2, W/2]$.

Before proceeding we must extend the usual definition of the LE, involving the taking of the limit $L \to \infty$, to finite length L. We consider a quasi-1D sample with the length L and width M ($L \gg M$). In the transverse direction we impose periodic boundary conditions. Our definition takes as its starting point the transfer matrix method of MacKinnon and Kramer.[4] We consider a transfer matrix T_L which is a product of a transfer matrix X_i for each slice up to the length L, $T_L = \prod_{i=1}^{L} X_i$. We prepare a random orthogonal $2M \times 2M$ matrix U_0. By repetition of a process involving several transfer matrix multiplication followed by a Gramm-Schmidt orthogonalization, we can express the matrix $T_L U_0$ as the product of an orthogonal matrix U_L and a right triangular matrix

$$T_L U_0 = U_L \begin{pmatrix} D_L^{(1)} & R_L^{(1,2)} & \cdots & R_L^{(1,2M)} \\ 0 & D_L^{(2)} & \ddots & R_L^{(2,2M)} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & D_L^{(2M)} \end{pmatrix} \tag{5}$$

We define the Lyapunov exponents $\gamma_L^{(i)}$ for finite length from the diagonal part of the right triangular matrix in (5)

$$\gamma_L^{(i)} = \frac{1}{L} \ln D_L^{(i)}. \tag{6}$$

In the present work, we concentrate on the statistics of the smallest positive Lyapunov exponent $\gamma_L = \gamma_L^{(M)}$. The LE for finite length L, defined in this way, is a random variable depending on the realisation of the random potential.

We study the dependence of the average $\langle \gamma_L \rangle$ and its variance σ^2 on the strength of disorder W, the energy E, the width M and the length L. The number of samples in each ensemble ranges from 1000 to 3000.

We have taken L sufficiently large that the average

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$\langle\gamma_L\rangle$ is independent of L. In this limit its value is equal to the standard Lyapunov exponent defined in the limit $L \to \infty$

$$\langle\gamma_L\rangle \simeq \gamma = \lim_{L\to\infty} \gamma_L. \tag{7}$$

Thus the inverse of $\langle\gamma_L\rangle$ is equal to the quasi-1D localisation length. Since, as is well known, this latter quantity obeys a one parameter scaling law we deduce that

$$\langle\gamma_L\rangle M = F_\gamma\left(\frac{\xi}{M}\right) \tag{8}$$

In the limit that $M \gg \xi$ we expect that

$$\langle\gamma_L\rangle M \to \frac{M}{\xi} \tag{9}$$

Thus it seems reasonable to approximate the scaling function (8) by the expansion

$$F_\gamma\left(\frac{\xi}{M}\right) = \frac{M}{\xi} + \sum_{n_\gamma=0}^{n_\gamma} a_n \left(\frac{\xi}{M}\right)^n. \tag{10}$$

Fitting our numerical data to this function, truncated at $n_\gamma = 1$, we obtain the localisation length ξ for each energy and disorder.

Next we consider the quantity $\sigma^2 L$. For small L, this depends on L. However, we restrict attention here to L sufficiently large that $\sigma^2 L$ becomes independent of L to within numerical accuracy. Since $\langle\gamma_L\rangle$ is also independent of L in this limit, a one parameter scaling relationship of the form (3) between the mean and variance of LE is possible. Our numerical data are consistent with $F_\sigma(\xi/M)$ approaching a constant value for large M/ξ. Given this an expansion of the form

$$F_\sigma\left(\frac{\xi}{M}\right) = \sum_{n_\sigma=0}^{n_\sigma} b_n \left(\frac{\xi}{M}\right)^n \tag{11}$$

is plausible. As none of the expansion coefficients is fixed, the absolute value of ξ and the fitting parameters b_n cannot be determined by fitting only to (11). To obtain their absolute values, we fix the localisation length at $E = 0.0$ and $W = 7.0$ as $\xi = 20.63$ according to the result of finite size scaling analysis of $\langle\gamma_L\rangle M$ presented above. After that we use (11), truncated at $n_\sigma = 4$, to find the absolute values of the coefficients b_n and the two-dimensional localisation length ξ for each value of energy and disorder.

The data and scaling function $F_\sigma(\xi/M)$ are shown in Fig. 1. It is seen that, within the accuracy of the simulation, all the data fall on a single curve confirming our assumption of a one parameter scaling for the variance as described by (3). The estimates of ξ obtained from the two analyses, based on the scaling of $\sigma^2 L/\langle\gamma_L\rangle$ and of $\langle\gamma_L\rangle M$, are in close agreement. This finding is strong evidence that the distribution of the LE in the two-dimensional Anderson model is described by a single parameter.

With decreasing M/ξ, $\sigma^2 L/\langle\gamma_L\rangle$ appears to approach unity consistent with the relation (1) for 1D. For large M/ξ, $\sigma^2 L/\langle\gamma_L\rangle$ approaches the asymptotic value b_0. We

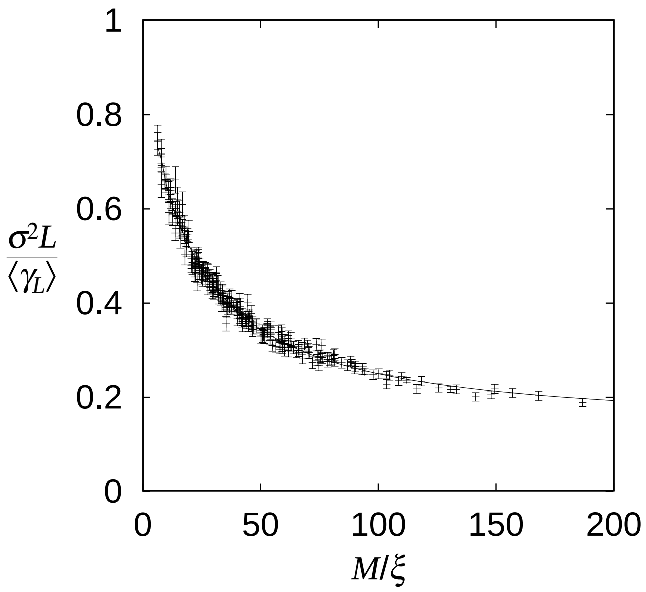

Fig. 1. $\sigma^2 L/\langle\gamma_L\rangle$ vs M/ξ. The solid line is the scaling function obtained by the analysis.

estimate

$$b_0 = \lim_{M/\xi\to\infty} F_\sigma\left(\frac{\xi}{M}\right) = 0.13 \pm .01. \tag{12}$$

This value is significantly smaller than the value of unity for one-dimensional systems indicating that the fluctuations in 2D systems are much weaker than in 1D systems.

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