

## Random path averaging in multiple-scattering theory

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A method of path averaging for waves propagating in a random dilute system of identical scatterers is developed. The scattering matrix of such a system is calculated. The method systematically takes into account repeating scatterings on the same scatterer and effects of correlations. Results obtained show the significant influence of new effects on the extinction of a coherent field and are valid in both the diffusive and localized regimes. [S0163-1829(97)04506-2]

### I. INTRODUCTION

Diffusion and localization of waves propagating in random media have attracted a great deal of interest within the last decade.<sup>1,2</sup> Even though considerable progress in understanding these phenomena has been achieved, many questions remain open. A commonly accepted picture is as follows. A coherent wave incident on a medium breaks into an incoherent stream of photons within the distance of the order of the photon scattering mean free path  $l$  which then travel randomly through the medium. Thus, one can expect a smooth diffusive distribution of the wave amplitude and strong fluctuations of the phase at space-time scales of the order of the photon mean free path and the free-flight time. When the wavelength,  $\lambda$ , is much smaller than  $l$ , the transport equation or the random walk approach give an adequate description of the photon diffusion in a medium.<sup>3</sup> On the other hand, when  $\lambda$  and  $l$  are comparable, wave interference can destroy ordinary diffusion and can lead to the wave localization. To incorporate interference effects into the theory of wave propagation in random media, one must start with the solution of the wave equation,

$$\{\omega^2[1 + \mathcal{E}(\mathbf{x})] + \Delta\}\Psi(\mathbf{x}) = 0. \quad (1)$$

Here  $\omega$  is the wave frequency and the function  $\mathcal{E}(\mathbf{x})$  describes properties of scatterers and their spatial distribution. When self-averaging of a field takes place in a random medium, observable quantities can be obtained upon averaging over an ensemble of random configurations of scatterers. An averaged Green's function of Eq. (1),  $\langle \hat{\mathbf{G}}_{\mathbf{x}} \rangle$ , defines the extinction length,  $l_{\text{ex}}$ , and the density of states. The transport properties of waves are described by the field-field correlation function,  $\langle \hat{\mathbf{G}}_{\mathbf{x}} \hat{\mathbf{G}}_{\mathbf{x}'}^* \rangle$ . With the help of the Dyson equation, analysis of  $\langle \hat{\mathbf{G}}_{\mathbf{x}} \rangle$  is usually reduced to calculations of the mass operator. The field-field correlator satisfies the Bethe-Salpeter equation. The latter involves the vertex function which describes an effective interaction of waves in a random medium. Microscopic analysis of these functions in a system of identical scatterers is based on multiple scattering perturbation theory. The perturbation procedure represents all quantities of interest as infinite series, where each term involves free-space Green's functions, one-particle scattering operators and, partial distribution functions of scatterers. Each term in the series corresponds to a particular sequence

of scattering events and has a multiplier  $n^m$ , where  $n$  is the average density of particles and  $m$  is the number of different scatterers participating in the scattering process. The theory contains three fundamental length scales, namely, the scatterer size,  $a$ , the separation between scatterers,  $n^{-1/3}$ , and the wavelength,  $\lambda$ . In the case of dilute systems, when the volume fraction of scatterers,  $na^3$ , is small, one can ignore all correlations in a scatterers distribution and use only the first-order terms of the mass operator and the vertex function as the leading terms of the expansions over small  $n$ . Most of the analytical results are based on this *leading approximation* in the low density limit and on simple corrections to it.

The Green's function and field-field correlator obtained in the leading approximation only take a restricted class of scattering diagrams into account. Only multiple scattering processes in which a wave scatters no more than once off any scatterer are included. This prescribes a "long memory" to a wave, such that when traveling in a medium it "remembers" all scatterers that were already visited and avoids them in its subsequent motion. In application to the wave diffusion, this restriction seems to contradict the underlying picture of randomly walking photons. In the case of particlelike photons this would lead to anomalous diffusion, significantly different from ordinary diffusion.<sup>4,5</sup> Besides, the fraction of these "long memory" paths is exponentially small for long path lengths. Therefore, the leading approximation must be accompanied by strong physical reasons for ignoring other statistically dominating scattering sequences. Such reasoning usually appeals to destructive interference of scattered waves which separates wave diffusion from diffusion of particles. This was analyzed in numerical calculations on finite clusters of scatterers,<sup>6</sup> but reliable analytical results in this area are absent. The problem arises because, in order to investigate an influence of wave interference, one must evaluate the entire series for the Green's function and the field-field correlator.

In this paper we propose a method that enables us to consider processes with repeated scattering in calculation of the scattering operator. Our method is based upon (i) averaging over random paths instead of averaging over ensemble configurations, (ii) the statistical independence of random jumps of scattered waves, and (iii) the existence of a hierarchy of length scales in dilute systems of scatterers.

### II. LOW DENSITY LEADING APPROXIMATION

The Green's function of Eq. (1) corresponds to a field induced by a  $\delta$  source. It satisfies an integral equation with

the iteration solution given by the formal operator series:

$$\hat{\mathbf{G}} = \hat{\mathbf{D}} + \hat{\mathbf{D}}\hat{\mathcal{E}}\hat{\mathbf{D}} + \hat{\mathbf{D}}\hat{\mathcal{E}}\hat{\mathbf{D}}\hat{\mathcal{E}}\hat{\mathbf{D}} + \dots = \hat{\mathbf{D}} + \hat{\mathbf{D}}\hat{\mathbf{T}}\hat{\mathbf{D}}, \quad (2)$$

where  $\hat{\mathbf{D}}$  is a free-space propagator and  $\hat{\mathbf{T}}$  denotes the total scattering operator of a system. One can choose any representation to obtain a constructive form of these expressions. For instance, in the coordinate representation, operators  $\hat{\mathbf{D}}$  and  $\hat{\mathcal{E}}$  have the following matrix elements:

$$\mathbf{D}_{\mathbf{xx}'} = \frac{e^{iq|\mathbf{x}-\mathbf{x}'|}}{4\pi|\mathbf{x}-\mathbf{x}'|}, \quad \mathcal{E}_{\mathbf{xx}'} = q^2\mathcal{E}(\mathbf{x})\delta_{\mathbf{xx}'}, \quad (3)$$

where  $q = \omega$  is a wave number in the system of units where speed of waves  $c = 1$ .

In the case of a single scatterer, using an exact solution of Eq. (1), one can obtain the scattering operator of a single particle,  $\hat{\mathbf{t}}$ . In a system of  $N$  identical scatterers, the total scattering operator can be expressed in terms of  $\hat{\mathbf{t}}$  operators:<sup>7</sup>

$$\hat{\mathbf{T}} = \sum_{s=1}^{\infty} \sum_{(\mathbf{x}_1 \dots \mathbf{x}_s)} \hat{\mathbf{t}}_{\mathbf{x}_s} \hat{\mathbf{D}} \dots \hat{\mathbf{D}} \hat{\mathbf{t}}_{\mathbf{x}_2} \hat{\mathbf{D}} \hat{\mathbf{t}}_{\mathbf{x}_1}, \quad (4)$$

where  $\hat{\mathbf{t}}_{\mathbf{x}_m}$  denotes the scattering operator of a particle located at the point  $\mathbf{x}_m$ . The sums in Eq. (4) account for all scattering paths,  $(\mathbf{x}_1 \rightarrow \mathbf{x}_2 \rightarrow \dots \rightarrow \mathbf{x}_s)$ , that are possible within a given spatial distribution of particles. Ensemble averaging of Eq. (4) eliminates this restriction and gives an average scattering operator  $\langle \hat{\mathbf{T}} \rangle_{\text{ensemble}} = \mathbf{T}$  in the form of the weighted sum of contributions from all scattering paths:

$$\mathbf{T} = \sum_{s=1}^{\infty} \int_{(\mathbf{x}_1 \dots \mathbf{x}_s)} W(\mathbf{x}_1 \dots \mathbf{x}_s) \hat{\mathbf{t}}_{\mathbf{x}_s} \hat{\mathbf{D}} \dots \hat{\mathbf{D}} \hat{\mathbf{t}}_{\mathbf{x}_2} \hat{\mathbf{D}} \hat{\mathbf{t}}_{\mathbf{x}_1}, \quad (5)$$

where the symbol  $\int_{(\mathbf{x}_1 \dots \mathbf{x}_s)}$  denotes integrations over all the indicated variables with the natural measure for each variable. The path weight,  $W(\mathbf{x}_1 \dots \mathbf{x}_s)$ , is the fraction of those configurations where the path is allowed and it can be expressed in terms of partial distribution functions of scatterers.

The conventional way of calculating  $\mathbf{T}$  in a statistically homogeneous system makes use of the momentum representation where the free-space propagator is diagonal,  $\mathbf{D}_{\mathbf{kk}'} = \delta_{\mathbf{kk}'} \mathbf{D}_k^\omega = \delta_{\mathbf{kk}'} [\mathbf{k}^2 - \omega^2]^{-1}$ , and the scattering matrix of a single particle depends upon the location of a scatterer in a simple way  $\langle \mathbf{k}' | \hat{\mathbf{t}}_{\mathbf{x}} | \mathbf{k} \rangle = \mathbf{t}_{\mathbf{k}'\mathbf{k}} \exp i\mathbf{x}(\mathbf{k}' - \mathbf{k})$ . Ignoring all short-range correlations arising from the rigid nature of scatterers, one can calculate exactly the ‘‘long memory’’ subseries. This subseries contains those terms of Eq. (5) where the scattered wave visits different scatterers only, so, all  $\mathbf{x}_m$  are independent and  $W(\mathbf{x}_1 \dots \mathbf{x}_s) = n^s$ . Retaining only these terms is known as the low density *leading approximation* or as the approximation of independent scattering. It is thought to be appropriate in the case of dilute systems of small scatterers and most of analytical results are based on it. For instance, the diagonal part of the  $\mathbf{T}$  operator in this approximation has the form<sup>8</sup>

$$\mathbf{T}_{\mathbf{k},\omega}^{\text{leading}} = n \mathbf{t}_{\mathbf{k},\omega} [1 - n \mathbf{t}_{\mathbf{k},\omega} \mathbf{D}_k^\omega]^{-1}, \quad (6)$$

where  $\mathbf{t}_{\mathbf{k},\omega} = \langle \mathbf{k} | \hat{\mathbf{t}} | \mathbf{k} \rangle$  is the scattering amplitude of a single particle in the forward direction.

However, even a naive statistical analysis gives rise to doubts regarding this approximation. Let us consider a system with the linear size  $L$  containing  $N$  scatterers. A wave propagates through the system by means of random jumps from one scatterer to another. During this motion a likely jump length is of the order of the mean free path  $l$ . In the diffusion limit, after  $s$  jumps, a wave covers the distance  $l\sqrt{s}$  and leaves the system when  $s \geq s_0 = L^2 l^{-2}$ . Therefore, paths with length of the order of  $s_0$  are most responsible for the diffusive content of an emerging field. A total number of paths consisting of  $s$  jumps is equal to  $N^s$ . The number of ‘‘long memory’’  $s$  paths is  $N!/(N-s)!$ . It is straightforward to estimate that in a macroscopic system the ratio of these two numbers, or fraction of the ‘‘long memory’’ paths, tends to zero as  $\exp[-\sqrt{s}/2n\sqrt{l}]$ . Thus, all ‘‘long memory’’ paths represent a statistically negligible fraction of the scattering paths that are important for the wave diffusion.

Moreover, the ‘‘long memory’’ in random walks provides an effect of ‘‘outward pressure’’ acting on a walker. For a simple walker the probability to stay within any volume around the origin after it makes  $s$  steps is proportional to  $N^s/V^s = n^s$ . When a walker is forced to avoid repeating scattering on the same scatterer, this probability is proportional to  $N!/[ (N-s)! s! V^s ] \approx n^s/s!$ . The longer paths we consider, the smaller, comparing with an unrestricted walker, this probability is. In the latter case an average distance from the origin,  $R_s$ , is of the order of  $l\sqrt{s}$ . Since for a particle with ‘‘long memory’’ the probability to stay within any volume around the origin is weakened by the factor  $1/s!$ , the  $R_s$  must be greater than  $l\sqrt{s}$ . This can be treated as an effective outward pressure that drives a particle much farther away from the origin. A similar pressure, caused by an excluded volume, exists in the case of the self-avoiding random walks and it leads to anomalous diffusion.<sup>4,5</sup>

From this point of view the success of the leading approximation in a description of the wave diffusion looks rather surprising. In any case, it is necessary to investigate the effect of all remaining paths on the scattering operator and the field-field correlator.

### III. RANDOM PATH AVERAGING

With the use of uncorrelated distributions of scatterers one can calculate the ‘‘long memory’’ subseries exactly. However, it does not help in the evaluation of the total scattering operator without bias against any scattering paths. To advance here we modify the averaging procedure. Averaging over random paths of scattered waves is physically equivalent to averaging over random distributions of scatterers. Therefore, we can rewrite Eq. (5) using jump vectors,  $\mathbf{R}_m = \mathbf{x}_{m+1} - \mathbf{x}_m$ , as independent variables instead of coordinates of scatterers

$$\mathbf{T} = \sum_{s=0}^{\infty} \int_{(\mathbf{x}_0, \mathbf{R}_1, \dots, \mathbf{R}_s)} \bar{W} \hat{\mathbf{t}}_{\mathbf{x}_0 + \mathbf{R}_1 + \dots + \mathbf{R}_s} \times \hat{\mathbf{D}} \dots \hat{\mathbf{D}} \hat{\mathbf{t}}_{\mathbf{x}_0 + \mathbf{R}_1 + \mathbf{R}_2} \hat{\mathbf{D}} \hat{\mathbf{t}}_{\mathbf{x}_0 + \mathbf{R}_1} \hat{\mathbf{D}} \hat{\mathbf{t}}_{\mathbf{x}_0}. \quad (7)$$

Any possible scattering path,  $\{\mathbf{x}_0, \mathbf{R}_1 \cdots \mathbf{R}_s\}$ , is accounted for in Eq. (7) with a path weight,  $\bar{W}(\mathbf{x}_0, \mathbf{R}_1 \cdots \mathbf{R}_s)$ . In the system of  $N$  scatterers fixed in space, the total number of  $s$  paths is equal to  $(N-1)^s$  and, as one can see from Eq. (4), they are all accounted for in  $\hat{\mathbf{T}}$ . When scatterers are randomly placed in space, a path weight is proportional to the probability to find all scatterers in proper places along a path. This probability is given by a joint distribution function of  $s$  scatterers which takes into account all correlations between them. If one ignores these correlations, a path weight takes a form used in the leading approximation,  $\bar{W} = [(N-1)/V]^s = n^s$ . Any attempt to preserve all correlations in  $\bar{W}$  presents too general problem with no constructive way to go far beyond the leading approximation. Meanwhile, a random path picture allows us to approach the problem utilizing the *random jump distribution* instead of distribution functions of scatterers. On each random step along the path  $\{\mathbf{x}_0, \mathbf{R}_1 \cdots \mathbf{R}_s\}$  a wave may jump to any of  $N-1$  scatterers. If we introduce a density of scatterers available for a single jump,  $ng(R)$ , and assume statistical independence of random steps, then the path weight can be factorized as follows:

$$\bar{W}(\mathbf{x}_0, \mathbf{R}_1 \cdots \mathbf{R}_s) = n^{s+1} g(R_1) \cdots g(R_s). \quad (8)$$

The function  $ng(r)$  describes the density of scatterers at a distance  $r$  from the scatterer placed at the origin. In a statistically homogeneous system this function is close to the average density,  $n$ . However, correlations between scatterers and fluctuations of their concentration cause deviations of  $g(r)$  from unity at small distances. This is supported by the following argument. The fluctuations of density are described by the Poisson distribution. In a dilute system an average volume per particle,  $n^{-1}$ , is much greater than the volume of a scatterer,  $a^3$ . Therefore, analyzing density fluctuations at large scales we can neglect the volume of the scatterers themselves. The probability to find  $m$  particles within a spherical shell of radius  $r$  and thickness  $n^{-1/3}$  is  $P(m, r) = (4\pi r^2 n^{2/3})^m \exp(-4\pi r^2 n^{2/3})/m!$ . It shows that the probability of large fluctuations of density rapidly decreases at  $r \gg n^{-1/3}$ . We can expect, therefore, that the deviations of  $g(r)$  from unity are substantial at distances comparable to the average separation between particles,  $n^{-1/3}$ . In application to wave propagation further simplification seems to be relevant. When the scattering cross section,  $\sigma_{sc}$ , is small the precise structure of  $ng(r)$  more likely can be disregarded since a wave with a large probability can just ignore a lot of surrounding scatterers. The probability that a wave makes an  $\mathbf{r}$  jump is proportional to the probability of not meeting any scatterer inside a cylinder of length  $r$  with the base area equal to  $\sigma_{sc}$ . From the Poisson distribution it follows that  $P(0, r\sigma_{sc}) = \exp(-nr\sigma_{sc})$ . Therefore, we can assume that the effect of density fluctuations on  $ng(r)$  is substantial at distances of the order of the scattering mean free path,  $(n\sigma_{sc})^{-1}$ . The function that takes into account the above arguments can be chosen in the form

$$ng(r) = n \left[ 1 - \frac{\gamma^3}{8\pi n} \exp(-r\gamma) \right]. \quad (9)$$

This function is normalized on  $N-1$  and, consequently, the path weight  $\bar{W}(\mathbf{x}_0, \mathbf{R}_1 \cdots \mathbf{R}_s)$  is normalized on the total num-

ber of  $s$  paths. A parameter  $\gamma$  is understood to be of the order of  $n\sigma_{sc}$ , if the scattering mean free path is much greater than  $n^{-1/3}$ , and it tends to the inverse average separation,  $n^{-1/3}$ , when the scattering cross section becomes large and both lengths become comparable.

In a dilute system Eq. (7) can be further simplified by noting that the size of a scatterer is much smaller than the average separation. If we use the coordinate representation in all intermediate states for the matrix elements  $\langle \mathbf{k}' | \hat{\mathbf{T}} | \mathbf{k} \rangle = \mathbf{T}_{\mathbf{k}'\mathbf{k}}$ ,

$$\begin{aligned} \mathbf{T}_{\mathbf{k}'\mathbf{k}} = & \sum_{s=0}^{\infty} \int_{(\mathbf{x}_0, \mathbf{R}_1 \cdots \mathbf{R}_s)} \bar{W} e^{i\mathbf{k}'(\mathbf{x}_0 + \mathbf{R}_s + \cdots + \mathbf{R}_1) - i\mathbf{k}\mathbf{x}_0} \\ & \times \int_{(\mathbf{r}_1, \mathbf{r}'_1 \cdots \mathbf{r}_s, \mathbf{r}'_s)} \mathbf{t}_{\mathbf{k}'\mathbf{r}'_s} \mathbf{D}_{\mathbf{r}'_s - \mathbf{r}_s + \mathbf{R}_s} \cdots \mathbf{D}_{\mathbf{r}'_1 - \mathbf{r}_1 + \mathbf{R}_1} \mathbf{t}_{\mathbf{r}_1 \mathbf{k}}, \end{aligned} \quad (10)$$

we can see that the ranges of the changes of variables  $(\mathbf{r}_1, \mathbf{r}'_1 \cdots \mathbf{r}_s, \mathbf{r}'_s)$  have a different order of magnitude. A single-particle scattering matrix in the coordinate representation,  $\mathbf{t}_{\mathbf{r}'\mathbf{r}}$ , vanishes outside the scatterer, where  $r', r \geq a$ , while the distances between two successive scatterers,  $R$ , are of the order of the average separation,  $n^{-1/3}$ . Assuming that short-range density fluctuations are not crucially important in the low density limit, we can assume that essential values of  $R$  are much greater than those of  $r', r$ . This fact allows one to use the far-zone asymptote of the free-space propagators in Eq. (10):

$$\mathbf{D}_{\mathbf{R} + \mathbf{r} - \mathbf{r}'} \approx \mathbf{D}_R e^{i\mathbf{q}(\mathbf{r} - \mathbf{r}')}, \quad \mathbf{D}_R = \frac{e^{iqR}}{4\pi R}, \quad (11)$$

where  $\mathbf{q} = \omega \hat{\mathbf{n}}, \hat{\mathbf{n}} = \mathbf{R}/R$ . It makes it possible to calculate all integrals with respect to  $(\mathbf{r}_1, \mathbf{r}'_1 \cdots \mathbf{r}_s, \mathbf{r}'_s)$  in Eq. (10):

$$\begin{aligned} \mathbf{T}_{\mathbf{k}'\mathbf{k}} \approx & \sum_{s=0}^{\infty} \int_{(\mathbf{x}, \mathbf{R}, \dots)} \bar{W} e^{i\mathbf{k}'(\mathbf{x}_0 + \mathbf{R}_s + \cdots + \mathbf{R}_1) - i\mathbf{k}\mathbf{x}_0} \\ & \times \mathbf{t}_{\mathbf{k}'\mathbf{q}_s} \cdots \mathbf{t}_{\mathbf{q}_1 \mathbf{k}} \mathbf{D}_{R_s} \cdots \mathbf{D}_{R_1}. \end{aligned} \quad (12)$$

In a statistically homogeneous system, the integral over the location of the first scatterer,  $\mathbf{x}_0$ , gives the common multiplier  $\delta_{\mathbf{k}'\mathbf{k}}$  expressing the homogeneity of the system. Since the distribution function,  $g(R)$ , depends upon the jump length only, the integrations over the directions of the jumps can be separated from the integrals over jump lengths in Eq. (12). All directional integrals can be evaluated with the help of the Taylor expansion of smooth functions in the integrand and a deformation of the integration contour in the complex plane:

$$\begin{aligned} \int_{\hat{\mathbf{n}}} \mathbf{t}_{\hat{\mathbf{n}}_1} \cdots \mathbf{t}_{\hat{\mathbf{n}}_s} \cdots e^{ikR\hat{\mathbf{n}}\hat{\mathbf{k}}} &= \int_0^{2\pi} d\phi \int_{-1}^1 d\mu f(\phi, \mu) e^{ikR\mu} \\ &\approx \sum_{\sigma=\pm} \frac{2\pi\sigma}{ikR} \mathbf{t}_{\sigma\omega\hat{\mathbf{k}}} \cdots \mathbf{t}_{\sigma\omega\hat{\mathbf{k}}} \cdots e^{ikR\sigma}. \end{aligned} \quad (13)$$

This formula works quite well in the case of weak scattering anisotropy,

$$\frac{\partial \mathbf{t}_{\dots \hat{\mathbf{n}}}}{\partial (\hat{\mathbf{n}} \cdot \hat{\mathbf{k}})} \ll kR \mathbf{t}_{\dots \hat{\mathbf{n}}}, \quad (14)$$

especially when  $kR \gg 1$ . Equation (13) becomes the rigorous equality for isotropic scattering. After utilizing Eq. (13), all the remaining integrals over jump lengths in Eq. (12) can be factorized since there is no correlations between jumps. The resulting expression for diagonal elements of  $\mathbf{T}_{\mathbf{k}'\mathbf{k}}$  takes the form

$$\mathbf{T}_{k\omega} = n \mathbf{t}_{\mathbf{k}\mathbf{k}} + n \sum_{s=1}^{\infty} \sum_{\sigma_1 \sigma_2 \dots} \mathbf{t}_{\mathbf{k}\sigma_s} g_{\sigma_s} \mathbf{t}_{\sigma_s \sigma_{s-1}} \dots g_{\sigma_2} \mathbf{t}_{\sigma_2 \sigma_1} g_{\sigma_1} \mathbf{t}_{\sigma_1 \mathbf{k}}. \quad (15)$$

In this equation the indexes  $\sigma_1, \sigma_2, \dots$  correspond to “+” or “-” and we denote

$$\mathbf{t}_{\mathbf{k}\sigma} = \langle \mathbf{k} | \mathbf{t} | \sigma \omega \hat{\mathbf{k}} \rangle, \quad \mathbf{t}_{\sigma \mathbf{k}} = \langle \sigma \omega \hat{\mathbf{k}} | \mathbf{t} | \mathbf{k} \rangle,$$

$$\mathbf{t}_{\sigma \sigma'} = \langle \sigma \omega \hat{\mathbf{k}} | \mathbf{t} | \sigma' \omega \hat{\mathbf{k}} \rangle, \quad g_{\sigma} = \frac{\sigma 2 \pi n}{ik} \int_0^{\infty} r g(r) \mathbf{D}_r^{\omega} e^{ikR\sigma} dr. \quad (16)$$

Due to the fact that all intermediate states in Eq. (15) are aligned along or against the incident momentum  $\mathbf{k}$ , the multiple scattering part of the  $\mathbf{T}$  matrix can be rewritten in terms of  $2 \times 2$  matrices  $\mathbf{t}$  and  $\mathbf{g}$  defined in Eq. (16):

$$\begin{aligned} \mathbf{T}_{k\omega}^{\text{multiple}} &= n \sum_{s=0}^{\infty} \sum_{\sigma \sigma'} \mathbf{t}_{\mathbf{k}\sigma} [\mathbf{g}(\mathbf{t}\mathbf{g})^s]_{\sigma \sigma'} \mathbf{t}_{\sigma' \mathbf{k}} \\ &= n \sum_{\sigma \sigma'} \mathbf{t}_{\mathbf{k}\sigma} [\mathbf{g}(1 - \mathbf{t}\mathbf{g})^{-1}]_{\sigma \sigma'} \mathbf{t}_{\sigma' \mathbf{k}}. \end{aligned} \quad (17)$$

Finally, using an explicit expression for the matrix  $(1 - \mathbf{t}\mathbf{g})^{-1}$ , we can represent the  $\mathbf{T}$  matrix in the form

$$\begin{aligned} \mathbf{T} &= n \mathbf{t}_{\mathbf{k}\mathbf{k}} + \frac{n}{1 - \text{tr}[\|\mathbf{t}\mathbf{g}\|] + \det[\|\mathbf{t}\mathbf{g}\|]} (\mathbf{t}_{+\mathbf{k}}, \mathbf{t}_{-\mathbf{k}}) \\ &\quad \times \begin{pmatrix} 1 - \mathbf{t}_{--} g_{-}, & \mathbf{t}_{-+} g_{+} \\ \mathbf{t}_{+-} g_{-}, & 1 - \mathbf{t}_{++} g_{+} \end{pmatrix} \begin{pmatrix} g_{+} \mathbf{t}_{\mathbf{k}+} \\ g_{-} \mathbf{t}_{\mathbf{k}-} \end{pmatrix}, \end{aligned} \quad (18)$$

where  $\text{tr}[\|\mathbf{t}\mathbf{g}\|]$  and  $\det[\|\mathbf{t}\mathbf{g}\|]$  denote the trace and the determinant, respectively. For a spherical scatterer  $\mathbf{t}_{++} = \mathbf{t}_{--}$  is the forward scattering amplitude and  $\mathbf{t}_{-+} = \mathbf{t}_{+-}$  is the back-scattering amplitude with all momenta confined at the “mass shell,”  $\mathbf{k}^2 = \omega^2$ , while in  $\mathbf{t}_{\mathbf{k}\pm}$  and  $\mathbf{t}_{\pm \mathbf{k}}$  one of the momenta ( $\mathbf{k}$ ) can be of an arbitrary magnitude.

This  $\mathbf{T}$  matrix is formally defined at all spatial scales and for any wavelength. In our evaluation, we assumed a weak anisotropy of scattering and a low density of particles,  $na^3 \ll 1$ . An isotropic scattering dominates, for instance, in the Rayleigh limit when a wavelength is much greater than the scatterer size,  $a\omega \ll 1$ . In this case, since scattering amplitudes do not depend upon directions of incoming and outgoing momenta, Eq. (18) can be simplified as follows:

$$\mathbf{T}_{k\omega} = n \left[ \mathbf{t}_{k\omega} + \frac{n \mathcal{T}_{k\omega}^2 (g_{+} + g_{-})}{1 - \mathbf{t}_{\omega} (g_{+} + g_{-})} \right], \quad (19)$$

where  $\mathbf{t}_{\omega} = \langle \omega \hat{\mathbf{k}} | \hat{\mathbf{t}} | \omega \hat{\mathbf{k}} \rangle$  is the on-shell scattering amplitude,  $\mathcal{T}_{k,\omega} = \langle \mathbf{k} | \hat{\mathbf{t}} | \omega \hat{\mathbf{k}} \rangle$  is the on-off-shell amplitude, and  $\mathbf{t}_{k,\omega} = \langle \mathbf{k} | \hat{\mathbf{t}} | \mathbf{k} \rangle$  is the off-shell one. All these quantities can be obtained from the solution of the boundary-value problem for a single scatterer and are well known for scatterers of a simple shape.<sup>9-11</sup> Quantities  $g_{\pm}$  are given in Eq. (16) and depend upon the choice of the density of surrounding scatterers. The substitution of the function  $g(r)$  given by Eq. (9) yields the total scattering operator of a dilute system of identical isotropic scatterers:

$$\mathbf{T}_{k\omega} = n \frac{\mathbf{t}_{k\omega} + n [\mathcal{T}_{k\omega}^2 - \mathbf{t}_{k\omega} \mathbf{t}_{\omega}] [\mathbf{D}_k^{\omega} - (\gamma^3 / 8 \pi n) \mathbf{D}_k^{\omega+i\gamma}]}{1 - n \mathbf{t}_{\omega} [\mathbf{D}_k^{\omega} - (\gamma^3 / 8 \pi n) \mathbf{D}_k^{\omega+i\gamma}]}. \quad (20)$$

This result involves the effective correlation radius  $\gamma^{-1}$ . In the limit  $\gamma \rightarrow \infty$ , when correlations between scatterers can be ignored, the  $\mathbf{T}$  matrix given by Eq. (20) reproduces the expression for the coherence length obtained in the leading approximation. The choice of  $\gamma \sim l_{\text{sc}}^{-1}$  physically means that the contribution to the coherent field from scattering paths consisting of many short jumps is assumed to be small, since they are rare in a statistically homogeneous dilute system.

#### IV. EXTINCTION LENGTH

The averaged Green’s function describes the coherent part of the field created by a point source placed in a random medium. Due to random scattering of propagating waves the pole of the Green’s function is shifted away from the pole of the free-space propagator,  $k = \omega$ . The imaginary part of this pole defines the extinction length of coherent waves in the random medium. According to the relationship between  $\mathbf{G}_{k,\omega}$  and the scattering matrix,  $\mathbf{G}_{k,\omega} = \mathbf{D}_k^{\omega} + (\mathbf{D}_k^{\omega})^2 \mathbf{T}_{k\omega}$ , it seems that  $\mathbf{G}_{k,\omega}$  has a pole of the second order when  $k$  tends to  $\omega$ . However, this singularity disappears, since  $\mathbf{T}_{k\omega}$  has there a zero of the first order,  $\mathbf{T}_{k\omega} = -2\omega(k - \omega) + \mathcal{O}[(k - \omega)^2]$ . Our result, as well as  $\mathbf{T}_{k\omega}^{\text{leading}}$  demonstrates this behavior. Therefore, only a pole in the  $\mathbf{T}$  matrix gives an actual singularity of the Green’s function. In the leading approximation this pole can be found from the equation

$$k^2 - \omega^2 = n \mathbf{t}_{k,\omega}. \quad (21)$$

This approximation relies on the low density of particles and weak scattering, which implies that  $f = (4\pi/3)a^3 n \ll 1$  and  $\omega \mathbf{t}_{k,\omega}$  is small. Weak scattering takes place in the Rayleigh limit where  $a\omega \ll 1$  and the scattering amplitude itself is a small quantity. An expansion over the small parameter  $a\omega$  yields the leading term of the scattering amplitude

$$\text{Re} \mathbf{t}_{k,\omega} \approx \varepsilon \frac{4}{3} \pi a^3 \omega^2, \quad (22)$$

where  $\varepsilon$  is the difference between dielectric constants of the scatterer and the host medium. The  $\text{Re} \mathbf{t}_{k,\omega}$  does not depend upon  $k$  and is of the second order, while the leading term of the imaginary part is of the fifth order, as it follows from the optical theorem:

$$\text{Im} \mathbf{t}_{k,\omega} = (2\pi)^2 \omega \int |\mathbf{t}_{\mathbf{k}\mathbf{k}'}|^2 d\hat{\mathbf{k}}'.$$

Solving Eq. (21) for long waves in a dilute system we obtain

$$k^2(\omega) = \omega \left[ 1 + f\varepsilon + i(f\varepsilon)^2 \frac{\omega^3}{4\pi n} \right].$$

When  $f\varepsilon \ll 1$ , this equation gives the well-known expression for the extinction length,

$$l_{\text{ex}}^{-1} = \text{Im} k(\omega) = n\sigma_{\text{sc}}/2 \approx \omega(f\varepsilon)^2 \omega^3/8\pi n.$$

Here the factor  $\omega^3/8\pi n$  is of the order of  $(R/\lambda)^3$ , where  $R$  and  $\lambda$  are the average separation between scatterers and the wavelength, respectively. It can be dangerously large, but in the weak scattering regime it is totally neutralized by the small parameter  $f\varepsilon$ . However, even in a rare mist of water drops this parameter can be of the order of unity due to the large value of  $\varepsilon$ . In similar dilute systems with large mismatch indexes the leading approximation still seems to be valid. For  $R \gg \lambda$  it gives

$$l_{\text{ex}}^{-1} \sim \omega f\varepsilon (R/\lambda)^{3/2} \gg \omega.$$

On the other hand, the extinction length (the phase coherence length) being much smaller than the wavelength is physically meaningless since then we cannot talk about wave propagation anymore. Therefore, the leading approximation requires that at least two conditions to be met, namely,  $a\omega \ll 1$  and  $f\varepsilon \ll 1$ .

Our  $\mathbf{T}$  matrix was obtained under the conditions  $f \ll 1$ ,  $a\omega \ll 1$ , and it involves a phenomenological parameter,  $\gamma = l_{\text{corr}}^{-1}$ . Our equation for the pole of the Green's function reads

$$[k^2 - \omega^2][k^2 - (\omega + i\gamma)^2] = n\mathbf{t}_\omega \left[ k^2 - (\omega + i\gamma)^2 - \frac{\gamma^3}{8\pi n} (k^2 - \omega^2) \right]. \quad (23)$$

To be consistent, we have to use the Rayleigh scattering amplitude given by Eq. (22). When  $f\varepsilon \ll 1$ , the right-hand part of Eq. (23) is small and we obtain

$$l_{\text{ex}}^{-1} \approx \gamma.$$

For  $f\varepsilon \sim 1$  and  $R \gg \lambda$ , taking into account that  $\lambda/l_{\text{corr}} \ll 1$  and  $\gamma^3/8\pi n \sim (R/l_{\text{corr}})^3 \sim 1$ , we have

$$l_{\text{ex}}^{-1} \approx \omega(f\varepsilon)^2 \frac{\omega^3}{8\pi n} \left( 1 - \frac{\gamma^3}{8\pi n} \right).$$

In both cases we obtain physically meaningful expressions for the extinction length since we have assumed that  $\gamma = l_{\text{corr}}^{-1}$  approaches  $n\sigma_{\text{sc}}$  in the weak scattering regime and it becomes proportional to  $n^{1/3}$  when the scattering cross section becomes large.

## V. CONCLUSIONS

In summary, we developed a new method for studying interference of multiple scattered waves that takes into account multiple returns to the same scatterer. We evaluated the scattering matrix of a dilute system of identical particles. The  $\mathbf{T}$  matrix obtained in Eq. (20) is valid in the Rayleigh limit,  $\omega a \ll 1$ . We did not impose any other restrictions on the wavelength, therefore, our result can be used to analyze the localized regime,  $\omega l \sim 1$ . All scattering paths and finite range of correlations are accounted for in our approach. This results in the momentum cutoff that appears in Eq. (20) as a truncation of the free-space propagator,  $\mathbf{D}_k^\omega$  by the term  $(\gamma^3/8\pi n)\mathbf{D}_k^{\omega+i\gamma}$ . It also gives an extra term in the numerator of Eq. (20), which is missing in  $\mathbf{T}^{\text{leading}}$ . In the Rayleigh limit this term is small because the leading terms of all scattering amplitudes are equal. However, the change of the denominator is physically important for the consistent description of the wave extinction in a system with large mismatch indexes. Using a microscopic expression for the density function,  $g(r)$ , in Eq. (19) one can obtain an exact expression for the extinction length. According to the definition of  $g(r)$ , this function must have the structure  $g(r) = 1 -$  ‘‘decay function.’’ It guarantees the appearance of the momentum cutoff in the denominator of the exact  $\mathbf{T}$  matrix. This effect is due to scattering paths with multiple returns. Even if each such path gives a negligible contribution, altogether they present a dominant amount of scattering paths and they cannot be neglected in the coherent field. Referring to the motivation of our work given in the Introduction, we can conclude that the interference cancellation of non-‘‘long-memory’’ contributions in the coherent field is overestimated in the leading approximation.

The  $\mathbf{T}$  matrix itself does not allow one to analyze the influence of ‘‘long memory’’ effects on transport properties of waves. Calculations of the field-field correlator are currently in progress. Our method enables us to proceed without utilizing the Bethe-Salpeter equation and does not require any additional approximations.

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