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Superoscillating response of a nonlinear system on a harmonic signal

D. G. Baranov[1,2] · A. P. Vinogradov[1,2,3] · A. A. Lisyansky[4,5]

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Abstract We demonstrate that a superoscillating in time signal may be obtained as a nonlinear response to a single low-frequency harmonic input. Using the realization of a superoscillating function proposed by Huang et al. (J Opt A Pure Appl Opt 9:S285, 2007), which is a mixture of five different harmonics, as an example, we synthesize the response function of such a nonlinear transformer and investigate its robustness with respect to the frequency and amplitude variations of the input signal.

Superoscillations are a counterintuitive mathematical effect, in which a band-limited function $f(t)$ (the function whose Fourier transform satisfies the condition $\hat{f}(\omega) = 0$ for all frequencies $|\omega| > \omega_{\max}$) may oscillate with a frequency much greater than $\omega_{\max}$. After the discovery of such functions by Berry [1], mathematical properties of superoscillating functions have been studied in detail [2–7], and various mathematical approaches to their construction have been suggested [8, 9]. The concept of superoscillations has proven to be extremely fruitful in nanophotonics, where superoscillations enable deep subwavelength focusing of electromagnetic fields without the use of evanescent waves [10–12] (for review see Ref. [13]). More recently, superoscillations have been studied in the time domain. Particularly, it has been shown that a quantum two-level emitter can be excited by a superoscillating electric field whose spectral components lie below the transition frequency of the emitter [14]. In another study, it has been found that a superoscillating electromagnetic signal can propagate through absorbing media over length scales far exceeding the absorption length [15].

Here, we explore a method of the nonlinear synthesis of a superoscillating signal from a low-frequency single harmonic input. We employ the technique of the harmonic synthesis [16] and explicitly construct the transformation function $f(z)$ that transforms a low-frequency harmonic $z(t)$ into a superoscillating function $y(t) = f(z(t))$.

Let us formulate the problem more rigorously. At the input of an inertialess nonlinear system, we have a harmonic oscillation $z(t) = \cos\omega_0 t$. The output function $y(t)$ is expected to show a superoscillating behavior. The problem is to find a nonlinear transformation function $f(z)$ that relates values of the input and output signals at the current time t. We are interested in the case when the output superoscillating function is represented as a superposition of N harmonic oscillations whose frequencies are multiple of ω_0. An example of such a superoscillating function is presented in Ref. [10]: $y(t) = \sum_{n=0}^{5} A_n \cos n\omega_0 t$, $A_0 = 1$, $A_1 = 13295000$, $A_2 = -30802818$, $A_3 = 26581909$, $A_4 = -10836909$, $A_5 = 1762818$, $\omega_0 = 1$. This function is plotted in Fig. 1, where the fastest harmonic $\cos(5t)$ is also shown for comparison. It is clearly seen that in the time interval $-0.1 < t < 0.1$, the superoscillating function $y(t)$ is well approximated by the function

✉ A. A. Lisyansky
alexander.lisyansky@qc.cuny.edu

[1] Moscow Institute of Physics and Technology, 9 Institutskiy per., Dolgoprudny, Russia 141700

[2] All-Russia Research Institute of Automatics, 22 Sushchevskaya, Moscow, Russia 127055

[3] Institute for Theoretical and Applied Electromagnetics, 13 Izhorskaya, Moscow, Russia 125412

[4] Department of Physics, Queens College of the City University of New York, Queens, NY 11367, USA

[5] The Graduate Center of the City University of New York, New York, NY 10016, USA

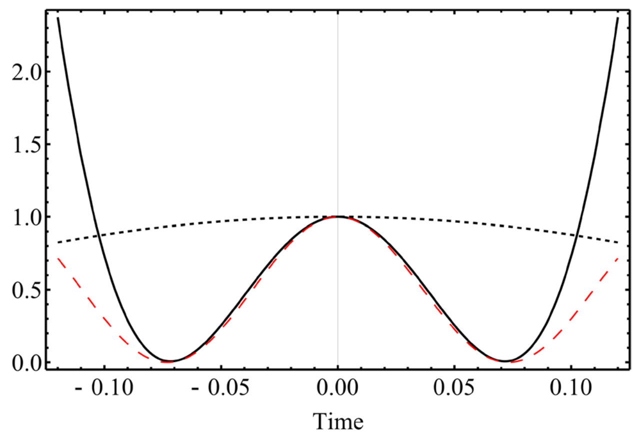

Fig. 1 Superoscillating function $y(t)$ containing spectral components $\omega_n = n$ (*solid line*) and the fastest component with frequency $\omega_5 = 5$ (*black dashed line*). The *red dashed line* shows the approximation of $y(t)$ near $t = 0$

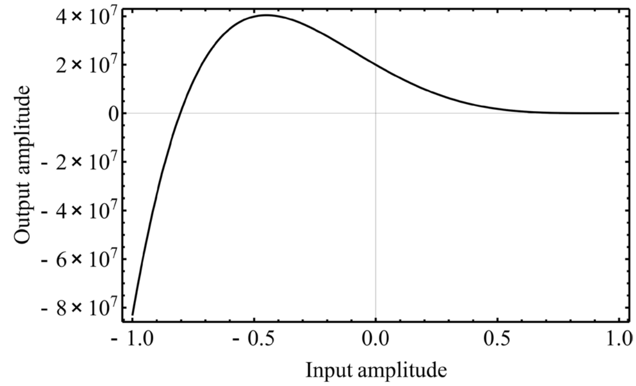

Fig. 2 Input–output characteristic of nonlinear transformation (4)

$f_{app}(t) = (\cos 43t + 1)/2$ that oscillates nearly nine times faster than the cutoff component.

We seek for the desired transformation function in the form of a polynomial:

$$f(z) = a_0 + a_1 z + \cdots + a_N z^N. \tag{1}$$

Then, the equality $y(t) = f(z(t))$ can be recast in the form

$$\sum_{n=0}^{N} a_n \cos^n \omega_0 t = \sum_{n=0}^{N} A_n \cos n\omega_0 t. \tag{2}$$

Having the set of values A_n that yields the superoscillating feature of $y(t)$, we can establish general expressions for coefficients a_n that constitute the transformation. Below, we show explicit expressions for the case of $N = 6$ harmonic components of the output signal, which can be found in Ref. [16]. It is convenient to write expressions for the odd and even coefficients separately:

$$\begin{aligned} a_0 &= 2^0(A_0 - A_2 + A_4), \\ a_2 &= 2^1(A_2 - 4A_4), \\ a_4 &= 2^3(A_4), \end{aligned} \tag{3a}$$

$$\begin{aligned} a_1 &= 2^0(A_1 - 3A_3 + 5A_5), \\ a_3 &= 2^2(A_3 - 5A_5), \\ a_5 &= 2^4(A_5). \end{aligned} \tag{3b}$$

These formulas can be generalized for a greater number of components using rules developed in Refs. [16, 17].

Equations (3a) and (3b) together with Eq. (1) determine the desired transformation function $f(z)$ of the nondelay system. The resulting transformation for the given superoscillating function takes the form:

$$\begin{aligned} f(z) = 10^7(&1.9965910 - 5.7636637z + 2.5089636z^2 \\ &+ 7.1071276z^3 - 8.6695272z^4 + 2.8205088z^5) \end{aligned} \tag{4}$$

Nonlinear characteristic of this transformation yielding superoscillating output is shown in Fig. 2. For the range of the input signal values $-1 \le z \le 1$, the output signal lies in the seven orders of magnitude broader range. However, such extreme amplification may be eliminated by scaling down the transformation by the factor of 10^7 [see Eq. (4)]. This decreases the superoscillations amplitude but leaves the shape of the superoscillating function unchanged. One of the features of the nonlinear transformation visible from Fig. 2 is that $f(0) \neq 0$, i.e., the current form of the transformation requires the nonzero response of the system to zero input. Obviously, this is an unphysical property of the system. To fix this, one can subtract the constant a_0 from the nonlinear transformation $f(z)$. Obviously, the shifted output $y(t) - a_0$ is still superoscillating, but the resulting system with the transformation $F(z) = f(z) - a_0$ does not generate any signal at zero input.

Now let us briefly discuss how robust the obtained transformation is against a variation of the input signal. Firstly, we note that transformation (4) is frequency scalable, i.e., if the input signal is given by $z_\alpha(t) = \cos \alpha\omega_0 t$, then one would obtain the output in the form $y_\alpha(t) = y(\alpha t)$, so that the superoscillating behavior of the output signal is preserved.

Figure 3 shows the output generated by the transformation $f(z)$ for different amplitudes z_0 of the input signal $z(t) = z_0 \cos \omega_0 t$. Due to its nonlinear character, transformation (4) distorts the signal when its amplitude differs from the reference value $z_0 = 1$ for which the initial transformation is designed by Eqs. (1–4). For the 1 % increase in the input amplitude, the output is still superoscillating, while the distance between the neighboring minima

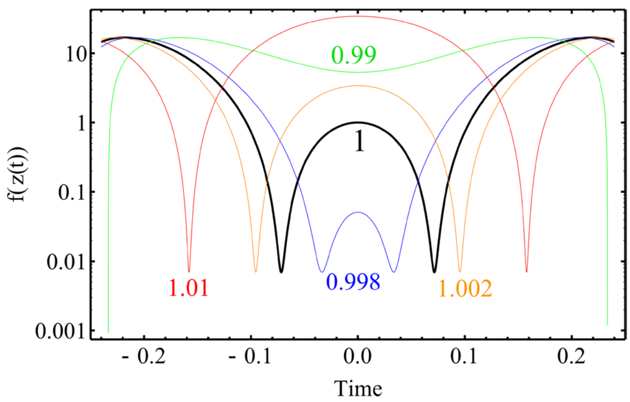

Fig. 3 Effect of variations of the input signal amplitude on the shape of the output signal $y(t)$. The *black thick curve* shows the output signal for the perfectly matched input amplitude $z_0 = 1$ also shown in Fig. 1. *Numbers* at the *curves* indicate the values of the input signal amplitude

increases (orange and red curves). The phenomenon is more sensitive; however, to a decrease in the input amplitude, its 1 % variation drastically changes the shape of the output function and kills superoscillations (the green curve). Overall, the current form of the nonlinear transformation is tolerant to ~0.5 % variation of the input signal amplitude.

The most straightforward way to implement the suggested nonlinear superoscillation synthesis is by using electronic components operating at radio frequencies. In the proposed system, the output signal y at the moment of time t is related to the input signal $z(t)$ at the same time. This implies that a physical system that performs the transformation $f(z)$ must be inertialess. There is a wide class of inertialess electronic components operating at radio frequencies that shows a nonlinear response [18]. The form of Eqs. (1) and (2) suggests that a system with the required transformation function can be realized with the use of frequency multipliers each of which transforms the input signal into a higher harmonic of the frequency $n\omega_0$, $n = 2, \dots, N$. Electronic circuits performing such multiplication were successfully demonstrated in configurations involving varactors [19] and nonlinear capacitors [20].

To conclude, we have demonstrated the synthesis of a superoscillating function from a single-frequency input signal in a generic nonlinear inertialess system. We have derived an expression for the system transformation function that performs such a synthesis. We have also discussed robustness of the superoscillation synthesis against variations of the input signal and potential experimental implementation of the proposed technique using nonlinear electronic components. This approach provides novel opportunities for controlling the signal transformation and frequency conversion in nonlinear systems. Furthermore, generated superoscillating signals may be used for probing other resonant systems with a frequency-dependent response.

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References

1. M.V. Berry, J. Phys. A: Math. Gen. **27**, L391 (1994)
2. A. Kempf, J. Math. Phys. **41**, 2360 (2000)
3. M.S. Calder, A. Kempf, J. Math. Phys. **46**, 012101 (2005)
4. P.J.S.G. Ferreira, A. Kempf, IEEE Trans. Signal Process. **54**, 3732 (2006)
5. A. Kempf, P.J.S.G. Ferreira, J. Phys. A: Math. Gen. **37**, 12067 (2004)
6. Y. Aharonov, F. Colombo, I. Sabadini, D.C. Struppa, J. Tollaksen, J. Phys. A: Math. Theor. **44**, 365304 (2011)
7. M.V. Berry, N. Moiseyev, J. Phys. A: Math. Theor. **47**, 315203 (2014)
8. P.J.S.G. Ferreira, A. Kempf, M.J.C.S. Reis, J. Phys. A: Math. Theor. **40**, 5141 (2007)
9. W. Qiao, J. Phys. A: Math. Gen. **29**, 2257 (1996)
10. F.M. Huang, Y. Chen, F.J. Garcia de Abajo, N.I. Zheludev, J. Opt. A: Pure Appl. Opt. **9**, S285 (2007)
11. F.M. Huang, N.I. Zheludev, Nano Lett. **9**, 1249 (2009)
12. E.T.F. Rogers, J. Lindberg, T. Roy, S. Savo, J.E. Chad, M.R. Dennis, N.I. Zheludev, Nat. Mater. **11**, 432 (2012)
13. E.T.F. Rogers, N.I. Zheludev, J. Opt. **15**, 094008 (2013)
14. D.G. Baranov, A.P. Vinogradov, A.A. Lisyansky, Opt. Lett. **39**, 6316 (2014)
15. Y. Eliezer, A. Bahabad, Opt. Express **22**, 31212 (2014)
16. A. M. Zaezdny, *Harmonic Synthesis in Radio Engineering and Telecommunications* (Energiya [in Russian], Leningrad, 1972)
17. A.M. Zaezdny, G.G. Men'shikov, USSR Comput. Math. Math. Phys. **6**, 234 (1966)
18. G.D. Vendelin, A.M. Pavio, U.L. Rohde, *Microwave Circuit Design Using Linear and Nonlinear Techniques* (Wiley, New York, 2005)
19. E. Bava, G.P. Bava, A. Godone, G. Rietto, IEEE Trans. Microw. Theory **27**, 141 (1979)
20. D.B. Leeson, S. Weinreb, Proc. IRE **47**, 1 (1959)